Transport equation with diffuse boundary condition.

$$\partial_t F + v\cdot\nabla_x F = 0 \qquad , \quad (t,x,v)\in\mathbb{R}_+\times\Omega\times\mathbb{R}^3 .$$

Ω: smooth and strictly convex.

$$F = \mu\int_{n\cdot v_1>0} F(t,x,v_1)\,(n(x)\cdot v_1)\,dv_1 \,. \qquad \iint_{\Omega\times\mathbb{R}^3} f(t,x,v)\,dx\,dv = \iint_{\Omega\times\mathbb{R}^3} f_0(x,v) = 0 .$$

Perturbation: $f(t,x,v) = F(t,x,v) - \mu(v)$.

Goal: under certain assumption on the initial data, $\exists!\, f$ s.t. $\sup_{t\geq 0}\|e^{\theta'|v|^2} f(t)\|_{L^\infty_{x,v}} \leq C\|e^{\theta'|v|^2} f_0\|_{L^\infty_{x,v}}$.

$\|f(t)\|_{L^1_{x,v}} \lesssim (\ln\langle t\rangle)^2\langle t\rangle^{-4} \times \{\text{initial}\}$

and $\sup_{x\in\Omega}\int_{\mathbb{R}^3} e^{\theta|v|^2}|f(t,x,v)|\,dv \leq C_0\langle t\rangle^{-3}(\ln\langle t\rangle)^2 \quad t\geq 0 .$

Decay of the L^1 norm

Lemma: Assume $f(0,x,v)\geq 0$, ~~suppose~~ $\exists\ m(x,v)\geq 0$ such that for $\forall\ T_0 \gg 1$, N,

$$f(NT_0,x,v) \geq m(x,v)\Big\{\iint_{\Omega\times\mathbb{R}^3} f((N-1)T_0,x,v)\,dx\,dv$$
$$-\iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{t_f(x,v)\geq\frac{3T_0}{4}} f((N-1)T_0,x,v)\,dv\,dx\Big\}.$$

Proof: iterate along characteristic

$$\Rightarrow \quad f(NT_0, x, v) \geq 1_{t_b(x,v) \leq \frac{T_0}{4}} \mu(v) \int_{V_1}\int_{V_2}\int_{V_3} 1_{t_3 \geq (N-1)T_0} \cdot f(t_3, x_3, v_3) \{n(x_3)\cdot v_3\} \, dv_3 \, d\sigma_2 \, d\sigma_1$$

change of variable

$$\geq \quad 1_{t_b(x,v) \leq \frac{T_0}{4}} \mu(v) \int_0^{t - t_b(x,v)}$$

$$\int_{\partial\Omega} \frac{|n(x_2)\cdot(x_1 - x_2)|}{|t_{b,1}|^4} \frac{|n(x_1)\cdot(x_1 - x_2)|}{t_{b,1}} \mu\left(\frac{|x_1 - x_2|}{t_{b,1}}\right)$$

$$\times \int_0^{t - t_b(x,v) - t_{b,1}} \int_{\partial\Omega} \frac{|n(x_3)\cdot(x_2 - x_3)|}{|t_{b,2}|^4} \frac{|n(x_2)\cdot(x_2 - x_3)|}{t_{b,2}} \mu\left(\frac{|x_2 - x_3|}{t_{b,2}}\right) \cdot$$

$$1_{t_3 \geq (N-1)T_0} \int_{n(x_3)\cdot v_3 > 0} f(t_3, x_3, v_3)(n(x_3)\cdot v_3) \, dv_3 \, dS_{x_3} \, dt_{b,2} \, dS_{x_2} \, dt_{b,1}.$$

Restrict range of x_2 as: $\delta > 0$

$$X_2^\delta = \{x_2 \in \partial\Omega : |x_1 - x_2| > \delta \text{ and } |x_2 - x_3| > \delta\}.$$

define $t_+ = t_{b,1} + t_{b,2} \in [0, T_0 - t_b(x,v)]$, $t_- = t_{b,1} - t_{b,2}$
$\in [-(T_0 - t_b(x,v)), T_0 - t_3]$

Introduce $A_+ = \{t_+ : T_0 - t_b - \min(t_b(x_3,v_3), \frac{T_0}{4}) \leq t_+ \leq T_0 - t_b(x,v)\}$.

$A_- := \{t_- : |t_-| \leq T_0 - t_b(x,v) - \min(t_b(x_3,v_3), \frac{T_0}{4})\}$

In (**): $\quad NT_0 - t_b(x,v) - t_f = T_0 - t_b(x,v) - t_f \in [0, \min(t_b(x_3,v_3), \frac{T_0}{4})]$

$$\Rightarrow (**) = \int_{T_0 - t_b - \min\{t_b(x_3,v_3), \frac{T_0}{4}\}}^{T_0 - t_b} f((N-1)T_0,\ x_3 - (T_0 - t_b(x,v) - t_f)v_3,\ v_3)\, dt_f$$

which implies $t_f(x_3 - (T_0 - t_b - t_f)v_3, v_3) \in [0, \frac{T_0}{4}]$

$$\Rightarrow (*) \geq 1_{t_b \leq \frac{T_0}{4}}\, C(\delta, T_0) \iint_{\Omega\times\mathbb{R}^3} 1_{t_f(y,v)\in[0,\frac{T_0}{4}]}\, f((N-1)T_0, y, v)\, dv\, dy$$

$$m(x,v) = 1_{t_b \leq \frac{T_0}{4}} \cdot C(\delta, T_0) = 1_{t_b \leq \frac{T_0}{4}} \frac{1}{(2\pi)^2} C_\Omega^4 \delta^8 T_0^{-9} \exp\left[-64\, \mathrm{diam}(\Omega)^2 T_0^{-2} / (x_2^\delta)\right] C_\mu \mu(v).$$

$\square$

Proposition: For $T_0 \gg 1$ and $\delta \ll 1$,

$$\|f(NT_0)\|_{L^1_{x,v}} \leq (1 - \|m\|_{L^1_{x,v}}) \|f((N-1)T_0)\|_{L^1_{x,v}}$$
$$+ 2\|m\|_{L^1_{x,v}} \|1_{t_f \geq \frac{T_0}{4}} f((N-1)T_0)\|_{L^1_{x,v}}.$$

$$\|m\|_{L^1_{x,v}} \approx C_{\delta, T_0}.$$

Proof:
$$f((N-1)T_0, x, v) = \overset{\text{positive}}{f_{N-1,+}(x,v)} - f_{N-1,-}(x,v)$$

①
$$= 1_{\cdot \geq 0}|\cdot| - 1_{\cdot < 0}|\cdot|$$

$f_\pm(s,x,v)$ solves eqn with initial data $f_{N-1,+}$ and $f_{N-1,-}$ at $s = (N-1)T_0$. Apply previous lemma to both $f_\pm$.

By $t_b(x,v) \le \frac{T_0}{4}$,

$$\min(t_{b,1}, t_{b,2}) = \min\left(\frac{t_+ + t_-}{2}, \frac{t_+ - t_-}{2}\right)$$

$$\ge \frac{1}{2}\left\{T_0 - t_b(x,v) - \frac{T_0}{4} - \frac{T_0}{4}\right\} \ge \frac{T_0}{8}.$$

$$\max\{t_{b,1}, t_{b,2}\} \le T_0$$

For $t_+ \in A_+$,

$$(N-1)T_0 \le t_3 = NT_0 - t_b - t_+ \le (N-1)T_0 + \min\left\{t_b(x_3,v_3), \frac{T_0}{4}\right\}.$$

$\Rightarrow$ if $t_f(y,v_3) = t_3 - (N-1)T_0 = T_0 - t_b - t_+ \in [0, \frac{T_0}{4}]$,

then $y = X((N-1)T_0; t_3, x_3, v_3)$, $\boxed{y \in \Omega}$.

$$(*) \ge \frac{C|x_i - x_{i+1}|^2}{T_0^4} \frac{|x_i - x_{i+1}|^2}{T_0} \frac{1}{2\pi} e^{-\frac{|x_i - x_{i+1}|^2}{2(T_0/8)^2}} \ge C(\delta, T_0).$$

$$\Rightarrow f(NT_0, x, v) \ge \mathbf{1}_{t_b(x,v) \le \frac{T_0}{4}} C(\delta, T_0)$$

$$\mu(v) \int_{\partial\Omega} dS_{x_3} \int_{n(x_3)\cdot v_3 > 0} dv_3 \{n(x_3)\cdot v_3\} \int_{A_+} dt_+$$

$$\cdot \int_{x_2^\delta} dS_{x_2} \int_{A_-} f(NT_0 - t_b(x,v) - t_+, x_3, v_3).$$

$$\ge \mathbf{1}_{t_b(x,v) \le \frac{T_0}{4}} C(\delta, T_0) \cdot \int_{\partial\Omega} dS_{x_3} \int_{n(x_3)\cdot v_3 > 0} dv_3 \{n(x_3)\cdot v_3\}$$

$$\cdot \int_{T_0 - t_b(x,v) - \min(t_b(x_3,v_3), \frac{T_0}{4})}^{T_0 - t_b(x,v)} dt_+ \, f(NT_0 - t_b(x,v) - t_+, x_3, v_3)$$

$(**)$

Conservation of mass:

$$\iint_{\Omega\times\mathbb{R}^3} f((N-1)T_0, x, v)\,dx\,dv = \iint_{\Omega\times\mathbb{R}^3} f_{N-1,+}(x,v) - f_{N-1,-} = 0$$

$$\Rightarrow \iint_{\Omega\times\mathbb{R}^3} f_{N-1,\pm}(x,v)\,dx\,dv = \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3} |f((N-1)T_0, x, v)|\,dx\,dv.$$

Then $f_\pm(NT_0, x, v) \geq m(x,v)\iint f_{N-1,\pm}(x,v)\,dx\,dv - m(x,v)\iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{t_f \geq \frac{T_0}{4}} f_{N-1,\pm}(x,v)\,dx\,dv$

$$:\geq I(x,v) := \frac{m(x,v)}{2}\iint_{\Omega\times\mathbb{R}^3} |f((N-1)T_0)| - m(x,v)\iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{t_f \geq \frac{T_0}{4}} |f((N-1)T_0)|$$

$$\Rightarrow |f(NT_0, x, v)| = |f_+(NT_0, x, v) - I(x,v) - f_-(NT_0, x, v) + I(x,v)|$$

$$\leq |f_+(NT_0, x, v) - I(x,v)| + |f_-(NT_0, x, v) - I(x,v)|$$

$$= \underbrace{f_+(NT_0, x, v) + f_-(NT_0, x, v)}_{\substack{\text{Solve eqn with} \\ \text{initial data } |f((N-1)T_0, x, v)|}} - 2I(x,v).$$

$$\Rightarrow \|f(NT_0)\|_{L^1_{x,v}} = \|f((N-1)T_0)\|_{L^1_{x,v}} - \|m\|_{L_{x,v}}\Big[\|f((N-1)T_0)\|_{L^1_{x,v}} - 2\|\mathbf{1}_{t_f \geq \frac{T_0}{4}} f((N-1)T_0)\|_{L^1_{x,v}}\Big]$$

$$\|m\|_{L^1_{x,v}} = \int_{\partial\Omega}\int_{n\cdot v>0}\int_{\max\{0, t_b - \frac{T_0}{4}\}}^{t_b(x,v)} \int_{n} \mu(u)|n\cdot v|\,ds\,dv\,dS_x.$$

$$\cancel{= \int_{\partial\Omega}\int_{n\cdot v>0}\Big(\mathbf{1}_{t_b\leq\frac{T_0}{4}}\int}$$

$\sim T_0|\partial\Omega|$ $\hookrightarrow$ canceled by $\frac{1}{T_0^n}$ $\square$.

Lemma: Suppose $\varphi(\tau) > 0$, $\varphi' > 0$ and.

$$\int_1^\infty \tau^{-5} \varphi(\tau)\, d\tau < \infty .$$

then $\|\varphi(t_f) f(t)\|_{L^1_{x,v}} + \int_{t^*}^{t} \|\varphi'(t_f) f\|_{L^1}$
$+ \int_{t^*}^{t} \varphi(t_f) f|_{L^1_{\gamma+}} - \frac{1}{4}\int_{t^*}^{t} |f|_{L^1_{\gamma+}} \leq \|\varphi(t_f) f(t^*)\|_{L^1_{x,v}}$
$+ C\|f(t^*)\|_{L^1_{x,v}}.$

Proof. In the sense of distribution

$$[\partial_t + v\cdot\nabla_x](\varphi(t_f)|f|) = \varphi'(t_f)\, v\cdot\nabla_x t_f |f| = -\varphi'(t_f)|f|$$

$\|\varphi(t_f) f(t)\|_{L^1} + \int_{t^*}^{t} \|\varphi(t_f) f(s)\|_{L^1} + \int_{t^*}^{t}\int_{\gamma+} \varphi(t_f)|f|\, dv\, dS_x .$
$\leq \|\varphi(t_f) f(t^*)\|_{L^1_{x,v}} + \int_{t^*}^{t}\int_{\partial\Omega}\int_{n\cdot v<0} \varphi(t_f)\, c_\mu \mu(v)\, |n(x)\cdot v| \cdot$
$\int_{n\cdot v_1>0} |f(s,x,v_1)|\, \{n(x)\cdot v_1\}\, dv_1 dv\, dS_x\, ds .$ (*).

Claim: ① $\sup_{x\in\partial\Omega} \int_{n\cdot v<0} \varphi(t_f)(x,v)\, c_\mu \mu(v) |n\cdot v|\, dv \lesssim 1.$

② $\int_{t^*}^{t} |f(s)|_{L^1_{\gamma+}} \leq \|f(t^*)\|_{L^1_{x,v}} + O(\delta^2) \int_{t^*}^{t} |f(s)|_{L^1_{\gamma+}}.$

then. $(*) \lesssim ① \int_{t^*}^{t}\int_{\gamma+} |f(s,x,v_1)|\{n(x)\cdot v_1\} \lesssim \|f(t^*)\|_{L^1} + \frac{1}{4}\int_{t^*}^{t} |f|_{L^1_{\gamma+}}$

Proof of Claim: ① split into $t_f \le \delta$ & $t_f > \delta$. $\left[\frac{n\cdot v}{|v|^2} = \frac{n\cdot(x_1-x_2)}{t_b|v|^2} \lesssim \frac{|x_1-x_2|^2}{|v|^2 t_b} \lesssim t_b\right]$

$$\int_{n(x)\cdot v<0} 1_{t_f\le\delta}\, y(t_f)\, r(x,v)\, \mu(v)\, |n\cdot v|\, dv$$

$$\lesssim \delta\, y(\delta) \int_{\mathbb{R}^3} |v|^3 \mu(v)\, dv \lesssim 1.$$

For $t_f > \delta$, change of variable.

$$\int_{n\cdot v<0} y(t_f)\, \mu(v)\, |n\cdot v|\, dv \lesssim \int_{\partial\Omega}\int_\delta^\infty y(t_f)\, \mu\left(\frac{|x-x_f|}{t_f}\right) \frac{|x-x_f|^4}{|t_f|^5}$$

$$\lesssim \int_\delta^\infty \frac{y(t_f)}{|t_f|^5}\, dt_f \lesssim 1.$$

② $\|f(t)\|_{L^1} \le \|f(t_*)\|_{L^1}$ from $\int_{t_*}^{t}\iint_{\gamma_+} |f| - \int_{t_*}^{t}\iint_{\gamma_-} |f| \ge 0$.

For $s \in (0, t-t_*)$, $|f(s,x,v)| \lesssim |f(t-\delta, x-(s-(t-\delta))v, v)|$ (A)

$+ |f(s-t_b(x,v), x_b(x,v), v)|$ (B).

t

s

$t-\delta$ $s-t_b$.

$(A) \lesssim \|f(t_*)\|_{L^1}$.

$$(B) = \int_{\gamma_+}\int_{t-\delta}^{t+t_b(x,v)} |f(s-t_b, x_b, v)|\, ds\; n\cdot v\, dS_x dv.$$

$$\le \int_{\partial\Omega}\int_{n(y)\cdot v<0} 1_{\delta > t_f} \int_{t-\delta}^{t} |f(s,y,v)|\, ds\, |n(y)\cdot v|\, dS_y\, dv.$$

$\hookrightarrow \lesssim \delta$ since $|n(y)\cdot v|/|v|^2 \lesssim \delta$. $\square$

Weight: $\varphi_0(T) = (\ln(e+1))^{-1} \ln(e+\ln(e+T))$,

$\varphi_1(T) = (e\ln(e+1))^{-1}(e+T)\ln(e+\ln(e+T))$ $\quad \varphi_1'(T) \geq (e\ln(e+1))^{-1}\varphi_0(T)$

$\varphi_3(T) = e^{-3}(T+e)^3 (\ln(T+e))^{-(1+\delta)}$,

$\varphi_4(T) = e^{-4}(T+e)^4 (\ln(T+e))^{-(1+\delta)}$. $\quad \varphi_4'(T) \geq \varphi_3(T)$

Satisfies condition in lemma. $\quad \varphi_i(0) = 1$.

Proposition:
$$\|f(NT_0)\|_{L^1} + \frac{4\delta C_{\delta,T_0}}{\varphi_{i-1}(\frac{T_0}{4})}\Big\{\|\varphi_{i-1}(t_f) f(NT_0)\|_{L^1}$$
$$+ \frac{1}{T_0}\|\varphi_i(t_f) f(NT_0)\|_{L^1} + \frac{1}{2T_0}\int_{(N-1)T_0}^{NT_0} |f|_{L^1_{\gamma+}}\Big\}$$
$$\leq \left(1 - \frac{C_{\delta,T_0}}{2}\right)\|f((N-1)T_0)\|_{L^1_{xv}} + \frac{4C_{\delta,T_0}}{\varphi_{i-1}(\frac{T_0}{4})}\Big\{\frac{1}{4}\|\varphi_{i-1}(t_f) f((N-1)T_0)\|_{L^1}$$
$$+ \frac{1}{T_0}\|\varphi_i(t_f) f((N-1)T_0)\|_{L^1}\Big\}$$

Proof: at γ_+, $\varphi(t_f) = \varphi(0) = 1$

$$\Rightarrow \int_{(N-1)T_0}^{NT_0} \|\varphi_i'(t_f) f\|_{L^1} \geq \int_{(N-1)T_0}^{NT_0} \|\varphi_{i-1}(t_f) f(t_*)\|_{L^1_{xv}}\, dt_*.$$

$$\geq T_0\|\varphi_{i-1}(t_f) f(NT_0)\|_{L^1} - CT_0\|f((N-1)T_0)\|_{L^1} \quad \text{(Lemma with } t_f=0\text{)}.$$

$$\Rightarrow \int_{(N-1)T_0}^{NT_0} \|\varphi_i(t_f) f(NT_0)\|_{L^1} + T_0\|\varphi_{i-1}(t_f) f(NT_0)\|_{L^1} + \frac{3}{4}\int_{(N-1)T_0}^{NT_0} |f|_{L^1_{\gamma+}}$$
$$\leq \|\varphi_i(t_f) f((N-1)T_0)\|_{L^1} + C(1+T_0)\|f((N-1)T_0)\|_{L^1}. \quad (**)$$

need to be controlled

$$1_{t_f \geq \frac{T_0}{4}} \leq (\varphi_{i-1}(\tfrac{T_0}{4}))^{-1} \varphi_{i-1}(t_f)$$

$$\|f(NT_0)\|_{L^1} \leq (1 - C_{\delta, T_0}) \|f((N-1)T_0)\|_{L^1} + 2C_{\delta, T_0} (\varphi_{i-1}(\tfrac{T_0}{4}))^{-1} (*).$$

$$(*) + \frac{4C_{\delta, T_0}}{T_0 \varphi_{i-1}(\frac{T_0}{4})} (**) \checkmark \qquad \|\varphi_{i-1}(t_f) f((N-1)T_0)\|_{L^1}.$$

L^1 decay: $\square$.

Proposition: $\|f(t)\|_{L^1} \lesssim C(\ln\langle t\rangle)^2 \langle t\rangle^{-4} \{ \|e^{\theta|v|^2} f_0\|_{L^\infty} + \|\varphi_4(t_f) f_0\|_{L^1} \}$.

Proof: $\|f\|_i = \|f\|_{L^1} + \frac{4C_{\delta, T_0}}{\varphi_{i-1}(\frac{T_0}{4})} \|\varphi_{i-1}(t_f) f\|_{L^1} + \frac{4C_{\delta, T_0}}{T_0 \varphi_{i-1}(\frac{T_0}{4})} \|\varphi_i(t_f) f\|_{L^1}$

$$\|f(NT_0)\|_i \leq \|f((N-1)T_0)\|_i \leq \cdots \leq \|f(0)\|_i .$$

$$\frac{\varphi_1}{\varphi_4} \text{ decreasing} \Rightarrow \varphi_1(t_f) = 1_{t_f \geq M} \varphi_1(t_f) + 1_{t_f < M} \varphi_1(t_f)$$

$$= 1_{t_f \geq M} \frac{\varphi_1(M)}{\varphi_4(M)} \varphi_4(t_f) + 1_{t_f < M} M \varphi_0(t_f).$$

$$\frac{1}{M} \|\varphi_1(t_f) f((N-1)T_0)\|_{L^1_{x,v}} \leq \frac{1}{M} \frac{\varphi_1(M)}{\varphi_4(M)} \|\varphi_4(t_f) f((N-1)T_0)\|_{L^1}$$

$$+ \|\varphi_0(t_f) f((N-1)T_0)\|_{L^1} \lesssim \frac{1}{M} \frac{\varphi_1(M)}{\varphi_4(M)} \frac{T_0 \varphi_3(\frac{T_0}{2})}{4C_{\delta, T_0}} \|f_0\|_4$$

$$+ \|\varphi_0(t_f) f((N-1)T_0)\|_{L^1_{x,v}} .$$

$$C_* = \max\left\{ (1 - C_{\frac{\delta_1 T_0}{2}}),\ (\tfrac{3}{4} + \tfrac{1}{T_0}),\ (1 - \tfrac{1}{M}) \right\}$$

$$\|f(NT_0)\|_1 \le C_* \|f((N-1)T_0)\|_1 + \underbrace{\frac{1}{M} \frac{\varphi_1(M)}{\varphi_4(M)} \frac{\varphi_3(\frac{T_0}{4})}{\varphi_0(\frac{T_0}{4})} \|f(0)\|_4}_{\to R}$$

$$\left(\frac{1}{T_0} \|\varphi_1(t) f((N-1)T_0)\|_{L^1} = \frac{1}{T_0}(1 - \tfrac{1}{M}) + \frac{1}{T_0 M} \|\cdots\|_{L^1} \le \frac{1}{T_0}(1 - \tfrac{1}{M}) + \cdots + \frac{1}{T_0} \right)$$

$(1 + \frac{1}{M})^{-1} \ge C_*$ (see later)

then $\|f(NT_0)\|_1 \le (1 + \frac{1}{M})^{-1} \|f((N-1)T_0)\|_1 + R$

$$\le (1 + \tfrac{1}{M})^{-N} \|f(0)\|_1 + (1+M)R$$

$$(1 + \tfrac{1}{M})^{-N} = \left((1 + \tfrac{1}{M})^{-M}\right)^{\frac{N}{M}} \le e^{-\frac{N}{2M}} \le e^{-\frac{t}{2T_0 M}}$$

$$(1+M)R \le 2 \frac{\varphi_1(M)}{\varphi_4(M)} \frac{\varphi_3(\frac{T_0}{4})}{\varphi_0(\frac{T_0}{4})} \|f(0)\|_4$$

$$\|f(t)\|_1 \lesssim \max\left\{ e^{-\frac{t}{2T_0 M}},\ \frac{\varphi_1(M)}{\varphi_4(M)} \right\} \left\{ \|f(0)\|_1 + \|f(0)\|_4 \right\}$$

Take $M = t\left[T_0 \ln(10+t)^3\right]^{-1}$ ✓ $\Rightarrow$ $\lesssim (\ln\langle t\rangle)^{2-\epsilon} \langle t\rangle^{-3}$. $\square$

$(t, z) \to P_t(z, E)$, transition probability. $S_t\mu(E) = \int \mu P_t(z, E)$

<u>Doeblin theorem</u>: Let $(S_t)_{t\geq 0}$ be a Markov semigroup $M(\Omega)$, defined on $M(\Omega)$ (space of finite measure) satisfying following condition:

$\exists\ \alpha \in (0,1)$, a probability measure η and some time $T>0$

s.t $S_T\mu \geq \alpha\eta$ for all $\mu \in P(\Omega)$

Then $(S_t)_{t\geq 0}$ has a unique invariant probability measure μ_∞ s.t. for any $\mu \in P(\Omega)$, $\|S_t\mu - \mu_\infty\|_1 \leq Ce^{-\lambda t}\|\mu - \mu_\infty\|_1$ for all $t\geq 0$